Very Weak, Essentially Undecidable Set Theories

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Δ_0-formulae

Definition

A Δ_0-formula is a set-theoretic formula where all the quantifiers are bounded, i.e., are of the form:

$$\begin{array}{lcl} (\forall x \in y)\varphi & \equiv & \forall x(x \in y \rightarrow \varphi) \\ (\exists x \in y)\varphi & \equiv & \exists x(x \in y \wedge \varphi) \end{array}$$

Definition

A Δ_0-formula is of complexity n if it can be rewritten through a Tarski-Mostowski computation in prenex form with $n-1$ quantifier alternations, starting with a universal quantifier.

(Essential) Undecidability

Definition

A set theory Θ is said to be UNDECIDABLE w.r.t. a class C of formulae if the satisfiability problem is unsolvable, i.e., if given a formula $\varphi \in C$, there is no algorithm that finds if there is a set assignment such that the formula holds.

Definition

A set theory Θ is ESSENTIALLY UNDECIDABLE if every one of its consistent recursively axiomatizable extensions is undecidable.

Gödel arguments for essential undecidability

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- we consider existential closures of classes C of Δ_0 formulae;
- a YES answer to the decision problem for φ implies $T \vdash \varphi^{\exists}$;
- a NO answer to the decision problem for φ implies $T \vdash \neg(\varphi^{\exists})$;

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- we consider existential closures of classes C of Δ_0 formulae;
- a $\mathrm{Y\scriptstyle ES}$ answer to the decision problem for φ implies $T \vdash \varphi^{\exists}$;
- a $\mathrm{N\scriptstyle O}$ answer to the decision problem for φ implies $T \vdash \neg(\varphi^{\exists})$;

Hence, in a consistent set theory T:

$$\text{Decidability wrt } C \implies \text{Completeness wrt } C$$

Our core theory

- Empty Set $\exists x \forall y \in x \; \neg y \in x$
- Adjunction $\forall x \forall y \exists z \forall w \big(w \in z \leftrightarrow (w = y \;\vee\; w \in x) \big)$
- Removal $\forall x \forall y \exists z \forall w \big(w \in z \leftrightarrow (\neg w = y \;\wedge\; w \in x) \big)$
- Regularity $\forall x \exists m \forall y \big(y \in x \rightarrow (m \in x \;\wedge\; \neg y \in m) \big)$

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We will also consider two extensions with:

- Separation for any φ, $\forall u \exists s \forall v \big(v \in s \leftrightarrow (v \in u \ \wedge \ \varphi)\big)$
- Finitude $\forall f (\forall t \in f)(\exists a \in f)(\forall b \in f)\big((\forall d \in b) d \in a \rightarrow b = a\big)$

Essential Undecidability in very weak foundational theories

- We already had essential undecidability w.r.t. $(\forall\exists\forall)_0$-formulae.
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Objectives:

- Express naturals and strongly represent total recursive functions;
- Find an encoding for formulae with a $(\forall\exists)_0$-definable total order;
- Define a $\mathsf{Proof}(x, y)$ predicate;
- Prove an analogous of the Fixpoint Theorem;
- Proceed with the standard Gödel arguments.

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$$(\exists\forall)_0$$

Pairs - 2

We opted for:

$$\begin{aligned} y @ x &:= \{ y \ \texttt{less} \ x \, , \ y \ \texttt{with} \ x \} \, , \\ \langle x, y \rangle &:= (x @ y) @ x \, . \end{aligned}$$

- Well suited for our axiomatic system;
- Projection extraction requires only existential quantifiers;
- No particular cases or exceptions;
- No pair is an ordinal.

Naturals with Axiom of Specification

Under the core with an instance of specification, we proved:

$$\begin{aligned}\mathrm{Num}(X) \quad \longleftrightarrow \quad & \forall\, y \in X^{+} \left(y = \varnothing \vee \exists\, z \in X \;\; z^{+} = y \right) \qquad \wedge \\ & (\forall\, u, v \in X \texttt{ less } \varnothing)\left(u \in v \vee v \in u \vee v = u \right),\end{aligned}$$

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$$(\forall\exists\forall)_0\,, \quad \text{as} \quad t = s^{+} \xleftrightarrow{\text{Def}} (\forall x \in t)(x = s \;\vee\; x \in s).$$

Naturals with Axiom of Specification

$$
\begin{aligned}
\mathsf{Num}(X) \overset{\text{Def}}{\longleftrightarrow}\ & \mathsf{Quadruple}(X) \wedge \\
& \mathsf{Fun}(\pi_4(X)) \wedge \pi_1(X) = \pi_2(X)^- \wedge \pi_3(X) = \pi_2(X)^+ \wedge \\
& (\forall n \in \pi_2(X))\mathsf{Triple}(\pi_4(X)(n)) \wedge \operatorname{dom} \pi_4(X) = \pi_2(X) \wedge \\
& (\forall t \in \operatorname{ran} \pi_4(X))(\pi_1(t) = \pi_2(t)^- \wedge \pi_3(t) = \pi_2(t)^+) \wedge \\
& (\forall u, v \in \pi_2(X) \mathtt{\ less\ } \varnothing)(u \in v \vee v \in u \vee u = v) \wedge \\
& (\forall y \in \pi_3(X))\big(y = \varnothing \vee (\exists z \in \pi_2(X))\pi_3((\pi_4(X))(z)) = y\big)
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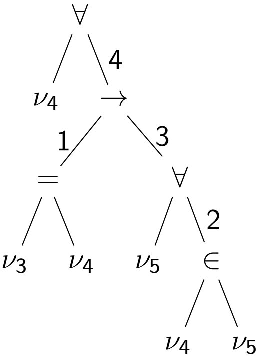
Codes

Take a set formula, e.g.:

$$\forall \nu_4 \big(\nu_3 = \nu_4 \rightarrow \forall \nu_5 (\nu_4 \in \nu_5) \big)$$



Function from $\{0, 1, 2, 3, 4, 5\}$ s.t.:

$$\begin{array}{lll} 0 & \mapsto & 7 \\ 1 & \mapsto & \langle =, 3, 4 \rangle \\ 2 & \mapsto & \langle \in, 4, 5 \rangle \\ 3 & \mapsto & \langle \forall, 5, 2 \rangle \\ 4 & \mapsto & \langle \rightarrow, 1, 3 \rangle \\ 5 & \mapsto & \langle \forall, 4, 4 \rangle \end{array}$$

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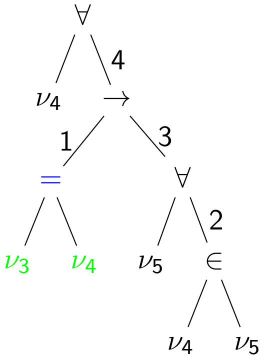

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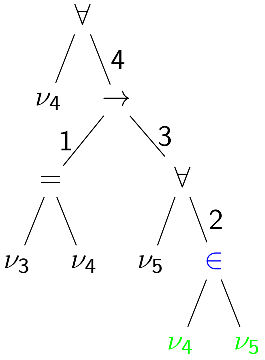

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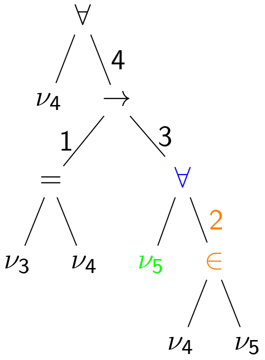

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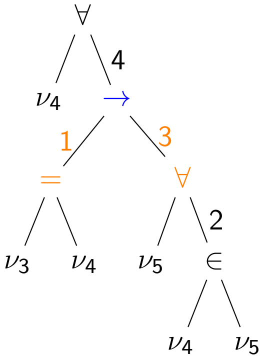

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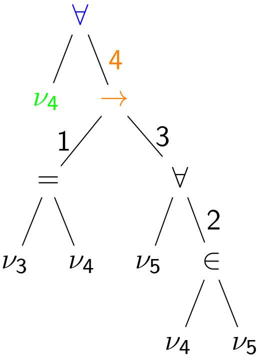

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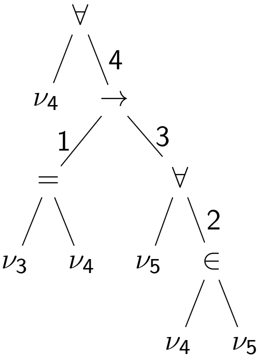

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$\mathsf{Cod}(x)$, $\leq_C$, and Next_C are $(\forall\exists)_0$.
With adjustments, so is $\mathsf{Form}(x)$.

Proofs

Several problems:

- Check if a variable is bound $\rightsquigarrow$ BoundList predicate;
- Check if two codes are equivalent $\rightsquigarrow$ CLCopy and CRCopy,
  - e.g. to check if modus ponens is applicable;
- Check if a formula is obtained from another through renaming;
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Again, we applied the technique of using functions/tuples to store in an easily accessible way complex information.

Proofs

Proofs are tuples containing:

- A list of all the subformulae of the formulae in the proof,
  - seen as triples containing clean left and right copies;
- A list of indices pointing to the first list (the proof sequence);
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It is possible to exploit the several parts to characterize all the rules, rename resolution, and the axioms.

By some technical considerations, this allows to prove the desired result.

Essential Undecidability - Novel results

Theorem

For every extension T' of T where natural numbers are sufficiently expressible through a $(\forall\exists)_0$ formula, i.e., every total recursive function on naturals is strongly representable, any Cod$-$*consistent recursively axiomatizable extension Θ of T' is undecidable with respect to $(\forall\exists)_0$ formulae.*

Corollary

T plus a single instance of the axiom schema of separation is essentially undecidable with respect to $(\forall\exists)_0$ formulae.

Corollary

T plus an axiom forcing the universe to be the one of hereditarily finite sets is essentially undecidable with respect to $(\forall\exists)_0$ formulae.

Some Future Developments

- Further refine the results applying similar techniques:
  - Find some minimal essentially undecidable theories wrt $(\forall\exists)_0$.
  - Finitude is restrictive.
  - Maybe the axiom of separation can be dropped.
- Generalize the techniques and tighten the class of formulae.
- Try to use a $(\forall\exists)_0$ characterization of the axiom of inifinity as a base for complexity reduction.

Thank you for your attention!

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Questions?